Spin Connections and Classification of Inequivalent Quantizations

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We discuss an extention of the quantization method based on the induced representation of the canonical group.

Quantum mechanics on a general configuration space was firstly studied by Dirac[1]. His method is the base of the quantum investigation of constraint systems. Next, Mackey[2] proposed another quantization method. He take a homogeneous space as the configuration space and he used the induced representation theory of group developed by Wigner[3]. The interesting point of his study is to show that there exist many inequivalent quantizations for general configuration space cases and that Dirac's method is one of them. But, Mackey's method is not so general, because the configuration space is the homogeneous space. Then, we will study the inequivalent quantization problem from more general viewpoint.

We consider a wavefunction as the most fundamental object in the quantum theory. Thus, we define firstly the wavefunction ($\psi(q)$) over the configuration space(Q) consistently and specify its properties including time-evolution. The wavefunction is not observed directly and a physical observable is a probability density ($\rho(q)$), which is defined as $\rho(q) \equiv \psi(q)^\dagger * \psi(q)$. The wavefunction is assumed to be a n-component complex valued function. Then, $\dagger$ means complex conjugate and $*$ is inner product.

Next we consider the time(t) evolution of the wavefunction. The probability density should satisfy the equation of continuity; $\frac{d}{dt}\rho(qt) = -\frac{\partial}{\partial\theta^a}J^a(qt)$, where θ^a is locally orthogonal coordinates. A new physical quantity $J^a(qt)$(probability current density) must be introduced for the probabilistic interpretation of the wavefunction. The form of the current must be determined and so we introduce the equation of the time evolution of the wavefuction, which is a linear first differential equation for time; $\frac{d}{dt}\psi(qt) = \hat{H}(q)\psi(qt)$ because of the probabilistic interpretation and of the principle of superposition. The forms of $J^a$and $\hat{H}$ are restricted to

$$\hat{H}(q) = i\frac{\partial}{\partial\theta^a}C(q)\frac{\partial}{\partial\theta^a} + V(q), \quad J^a(qt) = C(q)\frac{\partial}{\partial\theta^a}\psi(qt)^\dagger * \psi(qt) + C.C.$$

where $C(q)$ is a undetermined function. There exists another posibility (Dirac equation like) but the following arguments are not changed.

The inner product is written as $\rho(qt) \equiv \psi(qt)^{\mu\dagger}G_{\mu\nu}(q)\psi^\nu(qt)$, where $G_{\mu\nu}(q)$ should satisfy the properties of the metric. Thus, we can study the transformation properties of the wavefunction in imitation of Riemannian geometry. The correspondnce relations are $\vec{e}_\mu \equiv \frac{\partial}{\partial x^\mu} \iff \vec{E}_\mu(q),\ \vec{E}^\dagger_\mu(q), \quad \vec{V} = V^\mu\vec{e}_\mu \iff \psi(qt) = \vec{E}_\mu(q)\psi^\mu(qt), \quad (\vec{e}_\mu, \vec{e}_\nu) = g_{\mu\nu} \iff (\vec{E}^\dagger_\mu, \vec{E}_\nu) = G_{\mu\nu}(q)\ and\ (\vec{V}, \vec{U}) = V^\mu U^\nu g_{\mu\nu} \iff (\psi^\dagger, \psi) = \psi^{\dagger\mu}\psi^\nu G_{\mu\nu}$.

Now, we assume that in the locally orthogonal coordinate systems the metric of the wavefunction is Kronecker's delta. But there are many choices of the locally orthogonal coordinate systems and these systems are related each other by rotation. The wavefunction transforms with these rotations but physical observable (ρ) should be independent of the choices of the systems. Therefore, the wavefunction must be the base of the representation of the rotation group. This means the introduction of the spin degree.

Let us study the transformation property of the other physical observables. The definitions of the current and Hamiltonian include the derivative. Thus, we need to introduce the connection coefficients for the locally rotation. The basis vectors of the locally orthogonal coordinate system are dependent on a position. They

change under the transfor of position as $\nabla_{\vec{e}_a}\vec{e}_b(q) = \Gamma^c_{ab}\vec{e}_c(q)$ where Γ^c_{ab} is affine connection coefficients. Therefore, the basis vectors at a near point is given by $\vec{e}_b(q + d\theta^a\vec{e}_a) = (\delta_{bc} + d\theta^a\Gamma^c_{ab})\vec{e}_c(q)$. This part $(\delta_{bc} + d\theta^a\Gamma^c_{ab})$ means a infinitesimal rotation. Corresponding to this infinitesimal rotation, the base of the wavefunction transfors as $\vec{E}_\mu(q + d\theta^a\vec{e}_a) = V^\nu_\mu\vec{E}_\nu(q)$ where V^ν_μ is the unitary representaion of the infinitesimal rotation. Then, the spin connection is given by

$$\tilde{\nabla}_{\vec{e}_a}\vec{E}_\mu = Tr[\Gamma^c_{ab}(t^\alpha)^b_c]T^{\alpha\nu}_\mu\vec{E}_\nu$$

where t^α is an ajoint representaion of the so(d) Lie generator and T^α is the n dimentional representaion. We introduce an one-form $d\theta^a A^\alpha_a \equiv d\theta^a Tr[\Gamma^c_{ab}(t^\alpha)^b_c]$ which transforms as $A^\alpha T^\alpha \implies -V^\dagger dV + V^\dagger A^\alpha T^\alpha V$ under the local rotation v. Here V is the unitary representation matrix of v. The current $d\theta^a J_a = -\psi^{\dagger\mu}[\delta_{\mu\nu}d - A^\alpha T^\alpha_{\mu\nu}]\psi^\nu + C.C.$ is invariant under this rotation.

Next problem is to define the locally orthogonal coordinate systems over the manifold that is the configuration space Q. This problem is a pure mathematical problem. We introduce some charts and the locally coordinate are defined on each chart. In the overlap region two coordinate systems are consistently connected. That is, we define single group-valued function over the overlap region. When the overlap region is S^d, the single group-valued function are classifyed by the homotopy group. For examples, $\Pi_3(SO(d)) = Z$ and $\Pi_1(SO(2)) = Z$. Thus, the connection coefficients can be classifyed and it may be expected that quantization methods are done according to this claasification. But some cases of these classes may be unitary equivarent. We have not established this situation.

Let us apply our idea to the d-dimentional sphere cases. We immerse a sphere in R^{d+1} space, introduce two charts and use the stereographic projection. In each charts, basis vectors are given by $\vec{e}_a = \frac{1}{1+\vec{x}^2}\big(\delta_{ai}(1+\vec{x}^2) - 2x_a x_i,\ 2x_a\big)$ and $\vec{e}_a = \frac{1}{1+\vec{z}^2}\big(\delta_{ai}(1+\vec{z}^2) - 2z_a z_i,\ -2z_a\big)$, and the connection coefficients are

$$(S) \qquad \frac{1+\vec{x}^2}{2}\frac{\partial}{\partial x_a}\vec{e}_b = \Gamma^c_{ab}\vec{e}_c, \quad \Gamma^c_{ab} = (\delta_{ab}x_c - \delta_{ac}x_b)$$

$$(N) \qquad \frac{1+\vec{z}^2}{2}\frac{\partial}{\partial z_a}\vec{e}_b \;=\; \Gamma^c_{ab}\vec{e}_c, \quad \Gamma^c_{ab} \;=\; (\delta_{ab}z_c - \delta_{ac}z_b).$$

We can get the derivative of the wavefunction

$$(S) \;\; [\frac{1+\vec{x}^2}{2}\frac{\partial}{\partial x_a} \;-\; Tr[\Gamma^c_{ab}(t^\alpha)^b_c]T^\alpha]\psi, \quad (N) \;\; [\frac{1+\vec{z}^2}{2}\frac{\partial}{\partial z_a} \;-\; Tr[\Gamma^c_{ab}(t^\alpha)^b_c]T^\alpha]\psi.$$

Finally, for the $S^d(\simeq SO(d+1)/SO(d))$ case we show the relation between the induced representation approach[4,5] and ours. The derivative on the tangent space is constructed from the generators of $SO(d+1)$. This correspondence is

$$\sum_b \frac{r_b}{r^2}G_{ba} \iff \frac{1+\vec{x}^2}{2}\frac{\partial}{\partial x_a} \;-\; Tr[\Gamma^c_{ab}(t^\alpha)^b_c]T^\alpha$$

where G_{ba} is the generatotrs of SO(d+1) and $\vec{r}$ is the position vector on S^d in the R^{d+1} space.

References

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